

ASYMPTOTIC BEHAVIOR OF THE TOTAL LENGTH OF EXTERNAL BRANCHES FOR BETA COALESCENTS

JEAN-STÉPHANE DHERSIN AND LINGLONG YUAN

ABSTRACT. We consider a Λ -coalescent and we study the asymptotic behavior of the total length $L_{ext}^{(n)}$ of the external branches of the associated n -coalescent. For Kingman coalescent, i.e. $\Lambda = \delta_0$, the result is well known and is useful, together with the total length $L^{(n)}$, for Fu and Li's test of neutrality of mutations. For a large family of measures Λ , including Beta($2 - \alpha, \alpha$) with $0 < \alpha < 1$, Möhle has proved asymptotics of $L_{ext}^{(n)}$. Here we consider the case when the measure Λ is Beta($2 - \alpha, \alpha$), with $1 < \alpha < 2$. We prove that $n^{\alpha-2} L_{ext}^{(n)}$ converges in L^2 to $\alpha(\alpha - 1)\Gamma(\alpha)$. As a consequence, we get that $L_{ext}^{(n)}/L^{(n)}$ converges in probability to $2 - \alpha$. To prove the asymptotics of $L_{ext}^{(n)}$, we use a recursive construction of the n -coalescent by adding individuals one by one. Asymptotics of the distribution of d normalized external branch lengths and a related moment result are also given.

1. INTRODUCTION

1.1. Motivation. In a Wright-Fisher haploid population model with size N , we sample n individuals at present from the total population, and look backward to see the ancestral tree until we get to the most recent common ancestor (MRCA). If time is well rescaled and the size N of population becomes large, then the genealogy of the sample of size n converges weakly to the Kingman n -coalescent (see [25],[26]). During the evolution of the population, mutations may occur. We consider the infinite sites model introduced by Kimura [24]. In this model, each mutation is produced at a new site which is never seen before and will never be seen in the future. The neutrality of mutations means that all mutants are equally privileged by the environment. Under the infinite sites model, to detect or reject the neutrality when the genealogy is given by the Kingman coalescent, Fu and Li[17] have proposed a statistical test based on the total mutations numbers on the external branches and internal branches. Mutations happened on external branches affect only single individuals, so in practice they can be picked out easily according to the model setting. In this test, the asymptotics of the ratio $L_{ext}^{(n)}/L^{(n)}$ between the total length of the external branches $L_{ext}^{(n)}$ and the total length $L^{(n)}$ measures in some sense the weight of mutations happened on external branches among all.

For many populations, Kingman n -coalescent describes the genealogy quite well. But for some others, when descendants of one individual can occupy a big ratio of the next generation with non-negligible probability, it is no more relevant. It is for example the case of some marine species(see [1], [7], [15], [19], [21]). In this case, if time is well rescaled and the size of population becomes large, the ancestral tree converges weakly to the Λ n -coalescent,

Date: February 28, 2012.

2010 Mathematics Subject Classification. 60J70, 60J80, 60J25, 60F05, 92D25.

Key words and phrases. Coalescent process; Beta-coalescent; total length of external branches; block counting process; recursive construction; Fu and Li's statistical test .

Jean-Stéphane Dherisin and Linglong Yuan benefited from the support of the "Agence Nationale de la Recherche": ANR MANEGE (ANR-09-BLAN-0215).

where multiple collisions may appear. This class of coalescents has first been introduced by Pitman[29] and Sagitov[30]. Among Λ n -coalescent, a special and important subclass is called Beta n -coalescents characterized by Λ being a Beta distribution. The most popular ones are for the choice of parameters $2 - \alpha$ and α for $\alpha \in (0, 2)$.

$Beta(2 - \alpha, \alpha)$ n -coalescents arise not only in the context of biology. They also have connections with supercritical Galton-Watson process (see [31]), with continuous-stable branching processes (see [4]), with continuous random trees (see [2]). If $\alpha = 1$, we recover the Bolthausen-Sznitman n -coalescent which appears in the field of spin glasses(see [6], [8]) and is also connected to random recursive trees (see [20]). The Kingman coalescent is also obtained from the $Beta(2 - \alpha, \alpha)$ coalescent by letting α tend to 2.

For $Beta(2 - \alpha, \alpha)$ n -coalescents with $0 < \alpha < 2$, various quantities have been studied under some conditions on α , especially the law of the length of an external branch taken at random, the total length of external branches $L_{ext}^{(n)}$ and the total length $L^{(n)}$. A short survey is given in the next section. In this paper, we consider $Beta(2 - \alpha, \alpha)$ n -coalescent with $1 < \alpha < 2$. Asymptotics of the normalized length of an external branch taken at random have recently been proved by Dhersein *et al* in [11]. Here, we extend this result and give the asymptotics of the joint distribution of the lengths of a finite family of external branches taken at random, and use this result to get the asymptotics of $L_{ext}^{(n)}$. As a consequence, we get that the sequence $L_{ext}^{(n)}/L^{(n)}$ converges in probability to $2 - \alpha$.

1.2. Introduction and main results. Let \mathcal{E} be the set of partitions of $\mathbb{N} := \{1, 2, 3, \dots\}$ and, for $1 \leq j \leq n \leq \infty$, $\mathcal{E}_{j,n}$ the set of partitions of $\mathbb{N}_{j,n} := \{j, j+1, \dots, n\}$ if $n \in \mathbb{N}$ and $\mathbb{N}_{j,\infty} := \{j, j+1, \dots\}$. We denote by $\rho^{(j,n)}$ the natural restriction on $\mathcal{E}_{j,n}$: if $1 \leq l \leq j < n \leq m \leq \infty$ and $\pi = \{A_i\}_{i \in I}$ is a partition of $\mathbb{N}_{l,m}$, then $\rho^{(j,n)}\pi$ is the partition of $\mathbb{N}_{j,n}$ defined by $\rho^{(j,n)}\pi = \{A_i \cap \mathbb{N}_{j,n}\}_{i \in I}$. For a finite measure Λ on $[0, 1]$, we denote by $\Pi = (\Pi_t)_{t \geq 0}$ the Λ -coalescent process introduced independently by Pitman[29] and Sagitov[30]. The process $(\Pi_t)_{t \geq 0}$ is a càd-làg continuous time Markovian process taking values in \mathcal{E} with $\Pi_0 = \{\{1\}, \{2\}, \{3\}, \dots\}$ which is characterized by the càd-làg Λ n -coalescent processes $(\Pi_t^{(n)})_{t \geq 0} := (\rho^{(1,n)}\Pi_t)_{t \geq 0}$, $n \in \mathbb{N}$. For $n < m \leq \infty$, we have $(\Pi_t^{(n)})_{t \geq 0} = (\rho^{(1,n)}\Pi_t^{(m)})_{t \geq 0}$ (where $\Pi^{(\infty)} = \Pi$). This property is called the consistence property.

We introduce the measure $\nu(dx) = x^{-2}\Lambda(dx)$. For $2 \leq a \leq b$, we set

$$\lambda_{b,a} = \int_0^1 x^{a-2}(1-x)^{b-a}\Lambda(dx) = \int_0^1 x^a(1-x)^{b-a}\nu(dx).$$

For fixed $n \in \mathbb{N}$, we can think of $(\Pi_t^{(n)})_{t \geq 0}$ as a random tree with n leaves having labels from 1 to n . This is a Markovian process with values in \mathcal{E}_n , and its transition rates are given by: for $\xi, \eta \in \mathcal{E}_n$, $q_{\xi,\eta} = \lambda_{b,a}$ if η is obtained by merging a of the $b = |\xi|$ blocks of ξ and letting the $b - a$ others unchanged, and $q_{\xi,\eta} = 0$ otherwise. We say that a individuals (or blocks) of ξ have been coalesced in one single individual of η . Remark that the process $\Pi^{(n)}$ is an exchangeable process, which means that, for any permutation τ of $\mathbb{N}_{1,n}$, $\tau \circ \Pi^{(n)} \stackrel{d}{=} \Pi^{(n)}$.

The consistence property is equivalent to the equation:

$$(1) \quad \lambda_{b,a} = \lambda_{b+1,a+1} + \lambda_{b+1,a}.$$

This is Pitman's structure Theorem, see Lemma 18 in [29]. This relationship comes from the fact that a given merging blocks among b can coalesce in two ways while revealing an extra block : either the coalescence event implies the extra block (and then $a + 1$ blocks merge), either not.

Remark that $\Pi^{(n)}$ finally reaches one block. This final individual is called the most recent common ancestor (MRCA). We denote by $\tau^{(n)}$ the number of collisions it takes for the n individuals to be coalesced to the MRCA.

We define by $R^{(n)} = (R_t^{(n)})_{t \geq 0}$ the block counting process of $(\Pi_t^{(n)})_{t \geq 0}$: $R_t^{(n)} = |\Pi_t^{(n)}|$, which equals the number of blocks/individuals at time t . Then $R^{(n)}$ is also a continuous time Markovian process taking values in \mathbb{N}_n , decreasing from n to 1. At state b , for $a = 2, \dots, b$, each of the $\binom{b}{a}$ groups of a individuals coalesces independently at rate $\lambda_{b,a}$. Hence, the time the process $(R_t^{(n)})_{t \geq 0}$ stays at state b is exponential with parameter:

$$\begin{aligned} g_b &= \sum_{a=2}^b \binom{b}{a} \lambda_{b,a} \\ &= \int_0^1 \sum_{a=2}^b \binom{b}{a} x^a (1-x)^{b-a} \nu(dx) \\ &= \int_0^1 (1 - (1-x)^b - bx(1-x)^{b-1}) \nu(dx). \end{aligned}$$

We denote by $Y^{(n)} = (Y_k^{(n)})_{k \geq 1}$ the discrete time Markov chain associated with $R^{(n)}$. This is a decreasing process from $Y_0^{(n)} = n$ which reaches 1 at time $\tau^{(n)}$. The probability transitions of Markov chain $Y^{(n)}$ are given by: for $b \geq 2$ and $1 \leq l \leq b-1$,

$$\mathbb{P}(Y_k^{(n)} = b-l | Y_{k-1}^{(n)} = b) = \frac{\binom{b}{l+1} \lambda_{b,l+1}}{g_b},$$

and 1 is an absorbing state.

We introduce the discrete time process $X_k^{(n)} := Y_{k-1}^{(n)} - Y_k^{(n)}$, $k \geq 1$ with $X_0^{(n)} = 0$. This process counts the number of blocks we lose at the k -th jump. For $i \in \{1, \dots, n\}$ define

$$T_i^{(n)} := \inf \left\{ t \mid \{i\} \notin \Pi_t^{(n)} \right\}$$

as the length of the i th external branch and $T^{(n)}$ the length of a randomly chosen external branch. By exchangeability, $T_i^{(n)} \stackrel{d}{=} T^{(n)}$. We denote by $L_{ext}^{(n)} := \sum_{i=1}^n T_i^{(n)}$ the total length of the external branches of $\Pi^{(n)}$, and $L^{(n)}$ the total length of $\Pi^{(n)}$. Recall that the block counting process $R^{(n)}$ stays at state b an exponential time with parameter g_b . Hence we have

$$L^{(n)} = \sum_{k=0}^{\tau^{(n)}-1} \frac{Y_k^{(n)}}{g_{Y_k^{(n)}}} e_k,$$

where $e_k, k \geq 0$, are independent exponential random variables with parameter 1.

For several measures Λ , many asymptotic results on the external branches, their total length, and the total length of the Λ n -coalescent are already known.

- (1) If $\Lambda = \delta_0$, Dirac measure on 0, $\Pi^{(n)}$ is the Kingman n -coalescent. Then,
 - (a) $nT^{(n)}$ converges in distribution to T which is a random variable with density $f_T(x) = \frac{8}{(2+x)^3} \mathbf{1}_{x \geq 0}$ (See [5], [9], [22]).
 - (b) $L_{ext}^{(n)}$ converges in L^2 to 2 (see [17], [14]). A central limit theorem is also proved in [22].
 - (c) $\frac{L^{(n)}}{2} / \log n$ converges in probability to 1, and a central limit theorem is also known (see [10, 12]). Hence $(\log n) L_{ext}^{(n)} / L^{(n)}$ converges in probability to 1.

- (2) If Λ is the uniform probability measure on $[0, 1]$, $\Pi^{(n)}$ is the Bolthausen-Sznitman n -coalescent.
- (a) $(\log n)T^{(n)}$ is asymptotically standard exponentially distributed (see [16]).
 - (b) $\frac{\log n}{n}L^{(n)}$ converges in probability to 1 and a central limit theorem is also known (see [12, 13]).
- (3) If $\nu_{-1} = \int_0^1 x^{-1}\Lambda(dx) < \infty$, which includes the case of the $Beta(2-\alpha, \alpha)$ n -coalescent with $0 < \alpha < 1$,
- (a) $T^{(n)}$ converges in distribution to $Exp(\nu_{-1})$ (see [18, 28]).
 - (b) $L^{(n)}/n$ converges in distribution to a random variable L whose distribution coincides with that of $\int_0^\infty e^{-X_t}dt$, where X_t is a certain subordinator (see page 1405 in [12] and [27]).
 - (c) $L_{ext}^{(n)}/L^{(n)}$ converges in probability to 1 (see [28]).
- (4) If Λ is $Beta(2-\alpha, \alpha)$ with $1 < \alpha < 2$.
- (a) $n^{\alpha-1}T^{(n)}$ converges in distribution to T which is a random variable with density function $f_T(x) = \frac{1}{(\alpha-1)\Gamma(\alpha)}(1 + \frac{x}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}-1}\mathbf{1}_{x \geq 0}$ (see [11]).
 - (b) $n^{\alpha-2}L^{(n)}$ converges in probability to $\frac{\alpha(\alpha-1)}{2-\alpha}\Gamma(\alpha)$ (see [2]). A central limit theorem is also proved in [23] (see also [10]).

In the rest of the paper, we only consider $Beta(2-\alpha, \alpha)$ n -coalescents with $1 < \alpha < 2$. From now on, we omit $(2-\alpha, \alpha)$ and $1 < \alpha < 2$, and call it $Beta$ n -coalescent, or n -coalescent. The main result of the paper is the following asymptotic behavior of $L_{ext}^{(n)}$.

Theorem 1. *For $1 < \alpha < 2$, the following convergence in L^2 holds:*

$$n^{\alpha-2}L_{ext}^{(n)} \xrightarrow[n \rightarrow \infty]{L^2} \alpha(\alpha-1)\Gamma(\alpha).$$

Using this result and the asymptotics for $L^{(n)}$, we immediately get that

Corollary 2. *For $1 < \alpha < 2$, the following convergence in probability holds:*

$$\frac{L_{ext}^{(n)}}{L^{(n)}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2 - \alpha.$$

It shows that for $Beta$ n -coalescents with $1 < \alpha < 2$, the total length of the external branches and the total length are equivalent in the limit up to the multiplicative constant $2 - \alpha$. Recall that for Kingman n -coalescent, $(\log n)L_{ext}^{(n)}/L^{(n)}$ converges in probability to 1, and hence $L_{ext}^{(n)}/L^{(n)}$ converges in probability to 0. We know that $Beta$ n -coalescent converges weakly to Kingman n -coalescent as α tends to 2. This Corollary shows a nice continuity property for $\frac{L_{ext}^{(n)}}{L^{(n)}}$.

To prove Theorem 1, we first need the asymptotic behavior of the joint distribution of d external branch lengths.

Theorem 3. *For fixed $d \in \mathbb{N}$, and $T_1^{(n)}, T_2^{(n)}, \dots, T_d^{(n)}$ the lengths of the external branches of individuals $1, 2, \dots, d$, we have the following convergence in distribution:*

$$(n^{\alpha-1}T_1^{(n)}, n^{\alpha-1}T_2^{(n)}, \dots, n^{\alpha-1}T_d^{(n)}) \xrightarrow[n \rightarrow \infty]{(d)} (T_1, T_2, \dots, T_d)$$

where T_1, T_2, \dots, T_d are i.i.d variables with density function

$$f_T(x) = \frac{1}{(\alpha-1)\Gamma(\alpha)}(1 + \frac{x}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}-1}\mathbf{1}_{x \geq 0}.$$

This is an extension of Theorem 5.2 of [11] which is valid only for $d = 1$. This Theorem shows that the influence of each other individual among the d selected individuals is negligible face to the increasing number of other companions. In the limit, the existence for any multiple collision involving at least two of the d individuals has a negligible probability. When n tends to ∞ , each of the d selected individuals is asymptotically coalesced to the other $n - d$ individuals, independently of other $d - 1$ individuals.

To prove this, we use a recursive construction of the Λ n -coalescent. This construction has already been used in [11]. It is to add individuals one by one according to consistence. In our case, we add the chosen d individuals one by one to the rest of the $n - d$ individuals. The coalescence behavior of each individual can be investigated precisely which leads to Theorem 3.

The following Proposition gives a moment convergence for the normalized lengths of external branches.

Proposition 4. *Let T be a random variable with density function*

$$f_T(x) = \frac{1}{(\alpha - 1)\Gamma(\alpha)} \left(1 + \frac{x}{\alpha\Gamma(\alpha)}\right)^{-\frac{\alpha}{\alpha-1}-1} \mathbf{1}_{x \geq 0}.$$

(1) *Let β be a positive real number.*

(a) *If $\beta < \frac{\alpha}{\alpha-1}$, then $\lim_{n \rightarrow \infty} \mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta] = \mathbb{E}[T^\beta]$.*

(b) *If $\beta \geq \frac{\alpha}{\alpha-1}$, then $\lim_{n \rightarrow \infty} \mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta] = \infty$.*

(2) *For fixed $d \in \mathbb{N}$, and $\beta_1, \beta_2, \dots, \beta_d$ non-negative real numbers such that $\sum_{i=1}^d \beta_i < \frac{\alpha}{\alpha-1}$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta_1} (n^{\alpha-1}T_2^{(n)})^{\beta_2} \dots (n^{\alpha-1}T_d^{(n)})^{\beta_d}] = \mathbb{E}[T^{\beta_1}] \mathbb{E}[T^{\beta_2}] \dots \mathbb{E}[T^{\beta_d}].$$

Notice that the first part of the Proposition is suitable for any positive moment of $n^{\alpha-1}T_1^{(n)}$. The constant $\beta_0 = \frac{\alpha}{\alpha-1}$ appears as a critical point. For $\beta < \frac{\alpha}{\alpha-1}$, we have $\mathbb{E}[(T_1^{(n)})^\beta] \sim \mathbb{E}[T^\beta] n^{(1-\alpha)\beta}$, but for $\beta \geq \frac{\alpha}{\alpha-1}$, no equivalence could be found for $\mathbb{E}[(T^{(n)})^\beta]$. This becomes a curious point.

For the second part of the Proposition, we do not know whether $\beta_0 = \frac{\alpha}{\alpha-1}$ is a critical point or not.

Theorem 1 appears as a by-product of Theorem 3 and Proposition 4.

1.3. Organization of this paper. In Section 2, we describe the recursive construction. This construction is suitable for any Λ n -coalescent and is essential for the rest of the paper. In Section 3, we prove Proposition 4. Theorem 1 and Corollary 2 are proved in Section 4.

2. RECURSIVE CONSTRUCTION OF THE n -COALESCENT PROCESS

Recall that if $n \geq 3$ and $1 \leq i \leq j < n$, for any partition $\pi \in \mathcal{E}_{i,n}$, then $\rho^{(j,n)}\pi$ denotes the natural restriction of π to $\{j, j+1, \dots, n\}$. Let $n \geq 3$ and $1 \leq d \leq n-1$. Recall that if $\Pi^{(n)}$ is a n -coalescent, then $\rho^{(d+1,n)}\Pi^{(n)}$ is a $n-d$ -coalescent process on $\{d+1, \dots, n\}$. In this section, we consider $\Pi^{(d+1,n)}$, a realization of the $n-d$ -coalescent process on $\{d+1, \dots, n\}$, and we explain how to add successively individuals $d, d-1, \dots, 1$ to $\Pi^{(d+1,n)}$ to get $\Pi^{(d,n)}, \Pi^{(d-1,n)}, \dots, \Pi^{(1,n)} = \Pi^{(n)}$ such that $\rho^{(j,n)}\Pi^{(i,n)} = \Pi^{(j,n)}$, $1 \leq i \leq j \leq d+1$.

The above construction is called recursive construction. If $d = 1$, the construction is described in [11]. We recall it for self-contentedness.

2.1. The case $d = 1$. In this part, we consider $\Pi^{(2,n)}$ a coalescent on $\{2, \dots, n\}$ and we construct a coalescent $\Pi^{(1,n)} = \Pi^{(n)}$ on $\{1, 2, \dots, n\}$ such that $\rho^{(2,n)}\Pi^{(n)} = \Pi^{(2,n)}$ by sticking individual 1 to $\Pi^{(2,n)}$.

There are two possibilities to stick individual 1.

- Between two successive collisions of $\Pi^{(2,n)}$ (see Figure 1)

Let us call u and v the times of two successive collisions. We assume that individual 1 is not stuck at time u . For $t \in (u, v)$, the number of individuals of $\Pi_t^{(2,n)}$ (which is the number of its components) is constant. We call it b , and denote by i_1, i_2, \dots, i_b the individuals. Each of these b individuals attracts independently individual 1 at rate $\lambda_{b+1,2}$. That means that b independent exponential random variables e_k with parameter $\lambda_{b+1,2}$ are attached to the individuals, and if $\tau = u + e_{k_0} = \inf\{u + e_k, 1 \leq k \leq b\} < v$, then individual 1 is stuck to individual i_{k_0} (for the example in Figure 1, $i_{k_0} = \{2, 3\}$). Notice that individual 1 is coalesced at rate $b\lambda_{b+1,2}$. If it is stuck, then we define $\Pi^{(n)}$ by

- for $t < \tau$, the partition $\Pi_t^{(n)}$ is the partition $\Pi_t^{(2,n)}$ and the singleton $\{1\}$;
- for $t \geq \tau$, the partition $\Pi_t^{(n)}$ is obtained from the partition $\Pi_t^{(2,n)}$ by adding individual 1 to the block which has descendant i_{k_0} at time τ .

In this case, individual 1 is involved in a binary collision of $\Pi^{(n)}$.

- At a collision of $\Pi^{(2,n)}$ (see Figure 2)

We consider a collision of $\Pi^{(2,n)}$, say at a time u . Assume that individual 1 is not stuck before time u . We denote by $b = |\Pi_{u-}^{(2,n)}|$ the number of individuals just before the collision, and by k the number of individuals involved in this collision ($k = |\Pi_{u-}^{(2,n)}| - |\Pi_u^{(2,n)}| + 1$). Then we decide that individual 1 participates to this collision with probability

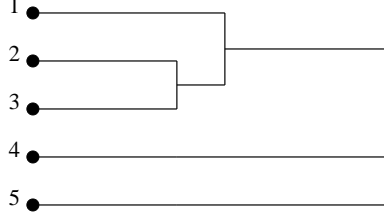
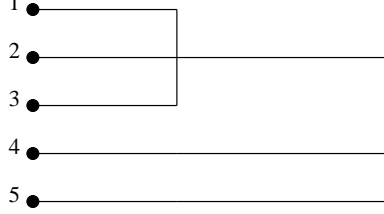
$$(2) \quad \frac{\lambda_{b+1,k+1}/g_b}{\lambda_{b,k}/g_b} = \frac{\int_0^1 x^{k+1}(1-x)^{b-k}\nu(dx)}{\int_0^1 x^k(1-x)^{b-k}\nu(dx)} = \frac{k-\alpha}{b}.$$

This equality comes from (1). If we decide that individual 1 participates to this collision, we set $\tau = u$ and stick 1 at this time to get $\Pi^{(n)}$, which means that $\Pi_t^{(n)}$ is defined as above on $[0, \tau)$ and $[\tau, +\infty)$. In this case, individual 1 is involved in a multiple collision of $\Pi^{(n)}$.

Remark that the length of the external branch of individual 1 in $\Pi^{(n)}$ is $T_1^{(n)} = \tau$. For the rest of the individuals, in general, the length of the external branch is not changed: for individual $2 \leq j \leq n$, we have $T_j^{(n)} = T_j(\Pi^{(2,n)})$, the length it had in $\Pi^{(2,n)}$. The only case it changes is when individual 1 is involved in a binary collision, and it is stuck to an individual i_{k_0} which is a singleton $\{j\}$. In that case, $T_j^{(n)} = \tau < T_j(\Pi^{(2,n)})$. All the other external lengths remain the same.

With this mechanism, we say that we have stuck individual 1 to $\Pi^{(2,n)}$ conditioning on $\Pi^{(2,n)}$ to get $\Pi^{(n)}$. Of course, $\rho^{(2,n)}\Pi^{(n)} = \Pi^{(2,n)}$.

2.2. The case $d > 1$. Using the recursive construction described in the previous case, given $\Pi^{(d+1,n)}$, we add one by one individuals $d, d-1, \dots, 1$ to get $\Pi^{(d,n)}, \Pi^{(d-1,n)}, \dots, \Pi^{(1,n)} = \Pi^{(n)}$. With this construction, we have $\rho^{(j,n)}\Pi^{(i,n)} = \Pi^{(j,n)}$ for all $1 \leq i \leq j \leq d+1$.

FIGURE 1. $n = 5$. Type 1: individual 1 encounters a binary collisionFIGURE 2. $n = 5$. Type 2: individual 1 encounters a multiple collision.

3. PROOF OF THEOREM 3:

Before proving Theorem 3, we give a technical Lemma. Recall that $\tau^{(n)}$ denotes the number of jumps of $\Pi^{(n)}$. For $1 \leq i \leq \tau^{(n)}$, we introduce $A_i^{(n)}$ the time at which the i -th jump of $\Pi^{(n)}$ occurs, and for $t \geq 0$, $r_t^{(n)} := \sup\{i, n^{\alpha-1}A_i^{(n)} \leq t\}$.

Lemma 1. *Let $(a_t)_{t \geq 0}$, $(b_t)_{t \geq 0}$ be two non-negative bounded continuous time stochastic processes and $(c_i)_{i \geq 1}$ a non-negative bounded discrete time process. Then, for fixed $0 \leq t_1 < t_2$, when n tends to ∞ :*

- (1) *the sequence $\exp(-\int_{t_1}^{t_2} \int_0^1 n^{1-\alpha}(R_{sn^{1-\alpha}}^{(n)} + a_s)x^2(1-x)^{R_{sn^{1-\alpha}}^{(n)}+b_s-1}\nu(dx)ds)$ converges in probability to $(\frac{\alpha\Gamma(\alpha)+t_2}{\alpha\Gamma(\alpha)+t_1})^{-\alpha}$;*
- (2) *the sequence $\prod_{i=r_{t_1}^{(n)}}^{r_{t_2}^{(n)}} \frac{Y_i^{(n)}+\alpha-1+c_i}{Y_{i-1}^{(n)}+c_i}$ converges in probability to $(\frac{\alpha\Gamma(\alpha)+t_2}{\alpha\Gamma(\alpha)+t_1})^{-\frac{\alpha(2-\alpha)}{\alpha-1}}$.*

Proof. For the first assertion, we refer to the arguments used in Theorem 5.2 in [11]. For the second assertion, we refer to the arguments used in Theorem 3.1 in [11]. \square

We are now able to give the proof of Theorem 3. Recall that $d \in \mathbb{N}$ is fixed, and $n \geq d+1$.

Let $\mathbb{R}_+^d := \{(x_1, \dots, x_d) : \forall i \in \{1, \dots, d\}, x_i \geq 0\}$. By Portmanteau Lemma (see [3], page 16), Theorem 3 is equivalent to: for any open set $\Omega_1 \subset \mathbb{R}_+^d$,

$$\liminf_{n \rightarrow \infty} \mathbb{P}((n^{\alpha-1}T_1^{(n)}, \dots, n^{\alpha-1}T_d^{(n)}) \in \Omega_1) \geq \mathbb{P}((T_1, \dots, T_d) \in \Omega_1).$$

We define $\Delta^{(d)} := \{(x_1, \dots, x_d) : 0 \leq x_d \leq x_{d-1} \leq \dots \leq x_1\}$. Notice that $(n^{\alpha-1}T_1^{(n)}, \dots, n^{\alpha-1}T_d^{(n)})$ and (T_1, \dots, T_d) are both exchangeable. Then Theorem 3 is equivalent to: for any open set $\Omega_2 \subset \Delta^{(d)}$,

$$(3) \quad \liminf_{n \rightarrow \infty} \mathbb{P}((n^{\alpha-1}T_1^{(n)}, \dots, n^{\alpha-1}T_d^{(n)}) \in \Omega_2) \geq \mathbb{P}((T_1, \dots, T_d) \in \Omega_2).$$

Remark that (T_1, \dots, T_d) has a continuous density function with respect to Lebesgue measure on $\Delta^{(d)}$. We denote it by f_d . For any measurable set $A \subset \Delta^{(d)}$, we define $Q(A) =$

$\int_A f_d(x)dx$. For any open set $\Omega_2 \subset \Delta^{(d)}$ with $Q(\Omega_2) > 0$ and $\epsilon > 0$, we can find a finite number of disjoint open rectangles $K_1, \dots, K_N, N \geq 1$ such that

$$\bigcup_{i=1}^N K_i \subset \Omega_2 \quad \text{and} \quad Q(\Omega_2) - \epsilon \leq \sum_{i=1}^N Q(K_i) \leq Q(\Omega_2).$$

Hence to prove (3), it suffices to get the result for any non-empty open rectangle $\Omega_2 \subset \Delta^{(d)}$, i.e. $\Omega_2 =]s_1, t_1[\times]s_2, t_2[\times \dots \times]s_d, t_d[$, where $0 < s_d < t_d < \dots < s_1 < t_1$. We shall prove that:

$$(4) \quad \lim_{n \rightarrow \infty} \mathbb{P}((n^{\alpha-1}T_1^{(n)}, \dots, n^{\alpha-1}T_d^{(n)}) \in \Omega_2) \geq \mathbb{P}((T_1, \dots, T_d) \in \Omega_2).$$

In fact we prove that if $\Omega_3 =]s_1, t_1[\times]s_2, t_2[\times \dots \times]s_d, t_d[$, where $0 < s_d < t_d < \dots < s_1 < t_1$,

$$(5) \quad \lim_{n \rightarrow \infty} \mathbb{P}((n^{\alpha-1}T_1^{(n)}, \dots, n^{\alpha-1}T_d^{(n)}) \in \Omega_3) = \mathbb{P}((T_1, \dots, T_d) \in \Omega_3).$$

Equality (5) directly implies inequality (4).

For simplicity, we first prove the case $d = 2$, and then explain the general case.

First step: case $d = 2$.

We consider $\Pi^{(3,n)}$, and construct $\Pi^{(n)}$ by adding at first individual 2, then adding individual 1, as explained in the previous Section. The event $\{s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1, s_2 < n^{\alpha-1}T_2^{(n)} \leq t_2\}$ is then the event where the time at which individual 2 is stuck is within $(n^{1-\alpha}s_2, n^{1-\alpha}t_2]$, and then individual 1 is stuck within $(n^{1-\alpha}s_1, n^{1-\alpha}t_1]$. Notice that when individual 2 is added, its external branch length is not changed by the coalescence of individual 1, and is equal to $T_2^{(n)}$. Let us write:

$$\begin{aligned} & \mathbb{P}(s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1, s_2 < n^{\alpha-1}T_2^{(n)} \leq t_2) \\ &= \mathbb{P}(s_2 < n^{\alpha-1}T_2^{(n)} \leq t_2) \mathbb{P}(s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1 | s_2 < n^{\alpha-1}T_2^{(n)} \leq t_2). \end{aligned}$$

By Theorem 5.2 in [11], $n^{\alpha-1}T^{(n)}$ converges in distribution to T which is a random variable with density function $f_T(x) = \frac{1}{(\alpha-1)\Gamma(\alpha)}(1 + \frac{x}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}-1}\mathbf{1}_{x \geq 0}$. So we get that $\mathbb{P}(s_2 < n^{\alpha-1}T_2^{(n)} \leq t_2)$ converges to

$$\mathbb{P}(s_2 < T \leq t_2) = (1 + \frac{s_2}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}} - (1 + \frac{t_2}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}}.$$

Hence it remains to prove that, when n tends to ∞ ,

$$(6) \quad \mathbb{P}(s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1 | s_2 < n^{\alpha-1}T_2^{(n)} \leq t_2) \rightarrow (1 + \frac{s_1}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}} - (1 + \frac{t_1}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}}.$$

Once individual 2 is stuck to $\Pi^{(3,n)}$, it gives $\Pi^{(2,n)}$. Since individual 1 is stuck to $\Pi^{(2,n)}$, we need to distinguish different behaviors of 2. As stated for the case $d = 1$, there are two possibilities for individual 2 to be stuck. The first is that individual 2 is involved in a binary collision within $(n^{1-\alpha}s_2, n^{1-\alpha}t_2]$. The second is that individual 2 is involved in a multiple collision of $\Pi^{(2,n)}$ within $(n^{1-\alpha}s_2, n^{1-\alpha}t_2]$.

We denote by $\Delta_1^{(3,n)}$ the first possibility, and by $\Delta_2^{(3,n)}$ the second possibility. Then $\{s_2 < n^{\alpha-1}T_2^{(n)} \leq t_2\} = \Delta_1^{(3,n)} \cup \Delta_2^{(3,n)}$, and

$$\begin{aligned}
& \mathbb{P}(s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1 | s_2 < n^{\alpha-1}T_2^{(n)} \leq t_2) \\
&= \mathbb{P}(s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1 | \Delta_1^{(3,n)} \cup \Delta_2^{(3,n)}) \\
&= \frac{\mathbb{P}(s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1, \Delta_1^{(3,n)} \cup \Delta_2^{(3,n)})}{\mathbb{P}(\Delta_1^{(3,n)} \cup \Delta_2^{(3,n)})} \\
&= \frac{\mathbb{P}(\Delta_1^{(3,n)})\mathbb{P}(s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1 | \Delta_1^{(3,n)}) + \mathbb{P}(\Delta_2^{(3,n)})\mathbb{P}(s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1 | \Delta_2^{(3,n)})}{\mathbb{P}(\Delta_1^{(3,n)} \cup \Delta_2^{(3,n)})}.
\end{aligned}$$

Hence to get (6) it is enough to prove that, when n tends to ∞ :

$$\begin{aligned}
(L_1) : \mathbb{P}(s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1 | \Delta_1^{(3,n)}) &\rightarrow (1 + \frac{s_1}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}} - (1 + \frac{t_1}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}}, \\
(L_2) : \mathbb{P}(s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1 | \Delta_2^{(3,n)}) &\rightarrow (1 + \frac{s_1}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}} - (1 + \frac{t_1}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}}.
\end{aligned}$$

Proof of (L_1) : On the event $\Delta_1^{(3,n)}$, we introduce i^*, t^* such that individual 2 is coalesced at time t^* and t^* is strictly between the $i^* - 1$ -th and i^* -th jump moment of $\Pi^{(3,n)}$. We denote by $\tau^{(3,n)}$ total jump times of $\Pi^{(3,n)}$. For $i \in \mathbb{N}$, we define $A_i^{(3,n)}$ the time of the i -th jump moment of $\Pi^{(3,n)}$ (hence $1 \leq i \leq \tau^{(3,n)}$), and for $t \geq 0$, $r_t^{(3,n)} := \sup\{i, n^{\alpha-1}A_i^{(3,n)} \leq t\}$.

Let $(a_t^{(3,n)})_{t \geq 0}$ be the continuous time stochastic process such that $a_t^{(3,n)} = \mathbf{1}_{[0, t^*)}(t)$, and $(b_i^{(3,n)})_{i \geq 1}$ be the discrete time stochastic process such that $b_i^{(3,n)} = 1$, if $1 \leq i \leq i^* - 1$, and $b_i^{(3,n)} = 0$ otherwise. Then

$$\begin{aligned}
& \mathbb{P}(s_1 < n^{\alpha-1}T_1^{(n)} \leq t_1 | \Delta_1^{(3,n)}) \\
&= \mathbb{E}[\exp(-\int_0^{s_1} n^{1-\alpha}(R_{sn^{1-\alpha}}^{(3,n)} + a_s^{(3,n)})\lambda_{R_{sn^{1-\alpha}}^{(3,n)} + a_s^{(3,n)} + 1, 2} ds) \\
& \quad \frac{Y_{i^*-1}^{(3,n)} + \alpha - 1}{Y_{i^*-1}^{(3,n)} + 1} \prod_{i=1}^{r_{s_1}^{(3,n)}} \frac{Y_{i-1}^{(3,n)} - X_i^{(3,n)} + b_i^{(3,n)} + \alpha - 1}{Y_{i-1}^{(3,n)} + b_i^{(3,n)}} \\
& \quad \cdot \left(1 - \exp(-\int_{s_1}^{t_1} n^{1-\alpha}(R_{sn^{1-\alpha}}^{(3,n)})\lambda_{R_{sn^{1-\alpha}}^{(3,n)} + 1, 2} ds) \prod_{i=r_{s_1}^{(3,n)}+1}^{r_{t_1}^{(3,n)}} \frac{Y_{i-1}^{(3,n)} - X_i^{(3,n)} + \alpha - 1}{Y_{i-1}^{(3,n)}}\right) | \Delta_1^{(3,n)}]
\end{aligned}$$

Recall that $\lambda_{b,a} = \int_0^1 x^a (1-x)^{b-a} \nu(dx)$ with $2 \leq a \leq b$. Then

$$\begin{aligned} & \mathbb{P}(s_1 < n^{\alpha-1} T_1^{(n)} \leq t_1 | \Delta_1^{(3,n)}) \\ &= \mathbb{E}[\exp(-\int_0^{s_1} \int_0^1 n^{1-\alpha} (R_{sn^{1-\alpha}}^{(3,n)} + a_s^{(3,n)}) x^2 (1-x)^{R_{sn^{1-\alpha}}^{(2,n)} + a_s^{(3,n)} - 1} \nu(dx) ds) \\ & \quad \frac{Y_{i^*-1}^{(3,n)} + \alpha - 1}{Y_{i^*-1}^{(3,n)} + 1} \prod_{i=1}^{r_{s_1}^{(3,n)}} \frac{Y_{i-1}^{(3,n)} - X_i^{(3,n)} + b_i^{(3,n)} + \alpha - 1}{Y_{i-1}^{(3,n)} + b_i^{(3,n)}} \\ & \quad \cdot \left(1 - \exp(-\int_{s_1}^{t_1} \int_0^1 n^{1-\alpha} (R_{sn^{1-\alpha}}^{(3,n)}) x^2 (1-x)^{R_{sn^{1-\alpha}}^{(2,n)} - 1} \nu(dx) ds) \right. \\ & \quad \left. \prod_{i=r_{s_1}^{(3,n)}+1}^{r_{t_1}^{(3,n)}} \frac{Y_{i-1}^{(3,n)} - X_i^{(3,n)} + \alpha - 1}{Y_{i-1}^{(3,n)}} \right) | \Delta_1^{(3,n)}]. \end{aligned}$$

Notice that

- (1) $\exp(-\int_0^{s_1} \int_0^1 n^{1-\alpha} (R_{sn^{1-\alpha}}^{(3,n)} + a_s^{(3,n)}) x^2 (1-x)^{R_{sn^{1-\alpha}}^{(2,n)} + a_s^{(3,n)} - 1} \nu(dx) ds)$ is the probability for individual 1 not to have a binary collision within $[0, n^{1-\alpha} s_1]$;
- (2) $\frac{Y_{i^*-1}^{(3,n)} + \alpha - 1}{Y_{i^*-1}^{(3,n)} + 1}$ is the probability that individual 1 is not involved in a collision which is

due to a binary collision between individual 2 and $\Pi^{(3,n)}$. Notice that $R^{(n-2)} \stackrel{d}{=} R^{(3,n)}$. By Theorem 4.4 in [11], $Y_{i^*-1}^{(3,n)} = R_{i^*}^{(3,n)} \geq R_{n^{1-\alpha} s_1}^{(3,n)} \rightarrow \infty$ when n tends to ∞ , so the probability $\frac{Y_{i^*-1}^{(3,n)} + \alpha - 1}{Y_{i^*-1}^{(3,n)} + 1}$ converges to 1;

- (3) $\prod_{i=1}^{r_{s_1}^{(3,n)}} \frac{Y_{i-1}^{(3,n)} - X_i^{(3,n)} + b_i^{(3,n)} + \alpha - 1}{Y_{i-1}^{(3,n)} + b_i^{(3,n)}}$ is the probability for individual 1 not to be coalesced in a multiple collisions of $\Pi^{(3,n)}$ within time $[0, n^{1-\alpha} s_1]$;

- (4) $1 - \exp(-\int_{s_1}^{t_1} \int_0^1 n^{1-\alpha} (R_{sn^{1-\alpha}}^{(3,n)}) x^2 (1-x)^{R_{sn^{1-\alpha}}^{(2,n)} - 1} \nu(dx) ds) \prod_{i=r_{s_1}^{(3,n)}+1}^{r_{t_1}^{(3,n)}} \frac{Y_{i-1}^{(3,n)} - X_i^{(3,n)} + \alpha - 1}{Y_{i-1}^{(3,n)}}$ is the probability for individual 1 to be coalesced within $(n^{1-\alpha} s_1, n^{1-\alpha} t_1]$.

Then using $R^{(n-2)} \stackrel{d}{=} R^{(3,n)}$ and Lemma 1, we get (L_1) .

Proof of (L_2) : On the event $\Delta_2^{(3,n)}$, we introduce t^{**} and i^{**} such that individual 2 is coalesced at time t^{**} which is the i^{**} -th jump moment of $\Pi^{(3,n)}$.

Let $(c^{(3,n)})_{t \geq 0}$ be the continuous time stochastic process such that $c_t^{(3,n)} = \mathbf{1}_{[0, t^{**})}(t)$, and $(d^{(3,n)})_{i \geq 1}$ and $(e^{(3,n)})_{i \geq 1}$ the discrete time stochastic processes such that if $1 \leq i \leq i^{**} - 1$, $d_i^{(3,n)} = e_i^{(3,n)} = 1$, if $i = i^{**}$, $d_i^{(3,n)} = 0$, $e_i^{(3,n)} = 1$, and $d_i^{(3,n)} = e_i^{(3,n)} = 0$ otherwise.

$$\begin{aligned}
& \mathbb{P}(\Delta^{(3,n)} | \Delta_2^{(3,n)}) \\
&= \mathbb{E}[\exp(-\int_0^{s_1} n^{1-\alpha} (R_{sn^{1-\alpha}}^{(3,n)} + c_s^{(3,n)}) \lambda_{R_{sn^{1-\alpha}}^{(3,n)} + c_s^{(3,n)} + 1, 2} ds) \\
& \quad \prod_{i=1}^{r_{s_1}^{(3,n)}} \frac{Y_{i-1}^{(3,n)} - X_i^{(3,n)} + d_i^{(3,n)} + \alpha - 1}{Y_{i-1}^{(3,n)} + e_i^{(3,n)}} \\
& \quad \cdot \left(1 - \exp(-\int_{s_1}^{t_1} n^{1-\alpha} R_{sn^{1-\alpha}}^{(3,n)} \lambda_{R_{sn^{1-\alpha}}^{(3,n)} + 1, 2} ds) \prod_{i=r_{s_1}^{(3,n)}+1}^{r_{t_1}^{(3,n)}} \frac{Y_{i-1}^{(3,n)} - X_i^{(3,n)} + \alpha - 1}{Y_{i-1}^{(3,n)}}\right) | \Delta_2^{(3,n)}] \\
&= \mathbb{E}[\exp(-\int_0^{s_1} \int_0^1 n^{1-\alpha} (R_{sn^{1-\alpha}}^{(3,n)} + c_s^{(3,n)}) x^2 (1-x)^{R_{sn^{1-\alpha}}^{(2,n)} + c_s^{(3,n)} - 1} \nu(dx) ds) \\
& \quad \prod_{i=1}^{r_{s_1}^{(3,n)}} \frac{Y_{i-1}^{(3,n)} - X_i^{(3,n)} + d_i^{(3,n)} + \alpha - 1}{Y_{i-1}^{(3,n)} + e_i^{(3,n)}} \\
& \quad \cdot \left(1 - \exp(-\int_{s_1}^{t_1} \int_0^1 n^{1-\alpha} (R_{sn^{1-\alpha}}^{(3,n)}) x^2 (1-x)^{R_{sn^{1-\alpha}}^{(2,n)} - 1} \nu(dx) ds) \right. \\
& \quad \left. \prod_{i=r_{s_1}^{(3,n)}+1}^{r_{t_1}^{(3,n)}} \frac{Y_{i-1}^{(3,n)} - X_i^{(3,n)} + \alpha - 1}{Y_{i-1}^{(3,n)}}\right) | \Delta_2^{(3,n)}].
\end{aligned}$$

Compared with (L_1) , in this case, no new collision is created by individual 2, and at the i^{**} -th collision of $\Pi^{(3,n)}$, which is also the collision where individual 2 is involved, the probability for individual 1 to be coalesced is $\frac{Y_{i^{**}-1}^{(3,n)} - X_{i^{**}}^{(3,n)} + \alpha - 1}{Y_{i^{**}-1}^{(3,n)} + 1}$.

Then using $R^{(n-2)} \stackrel{d}{=} R^{(3,n)}$ and Lemma 1, we get (L_2) . So we get the case $d = 2$.

Second step: $d \geq 3$.

The event $\{s_d < n^{\alpha-1} T_d^{(n)} \leq t_d, \dots, s_1 < n^{\alpha-1} T_1^{(n)} \leq t_1\}$ can also be interpreted as above. Given $\Pi^{(d+1,n)}$, we first add individual d which has to be stuck at a time within $(n^{1-\alpha} s_d, n^{1-\alpha} t_d]$, then we add individual $d-1$ which has to be stuck within $(n^{1-\alpha} s_{d-1}, n^{1-\alpha} t_{d-1}]$, etc. Since every individual is coalesced in a different interval, the external branch length of individual i when it is initially connected is exactly $T_i^{(n)}$, $1 \leq i \leq d$. We can then consider $T_i^{(n)}$ as the external branch length of individual i when it is initially added.

We use the decomposition

$$\begin{aligned}
& \mathbb{P}(s_d < n^{\alpha-1} T_d^{(n)} \leq t_d, \dots, s_1 < n^{\alpha-1} T_1^{(n)} \leq t_1) \\
&= \mathbb{P}(s_d < n^{\alpha-1} T_d^{(n)} \leq t_d) \mathbb{P}(s_{d-1} < n^{\alpha-1} T_{d-1}^{(n)} \leq t_{d-1} | s_d < n^{\alpha-1} T_d^{(n)} \leq t_d) \\
& \quad \dots \mathbb{P}(s_1 < n^{\alpha-1} T_1^{(n)} \leq t_1 | s_2 < n^{\alpha-1} T_2^{(n)} \leq t_2, \dots, s_d < n^{\alpha-1} T_d^{(n)} \leq t_d).
\end{aligned}$$

Notice that by Theorem 5.2 in [11], $\mathbb{P}(s_d < n^{\alpha-1}T_d^{(n)} \leq t_d)$ converges to $(1 + \frac{s_d}{\alpha\Gamma(\alpha)})^{-\frac{1}{\alpha-1}} - (1 + \frac{t_d}{\alpha\Gamma(\alpha)})^{-\frac{1}{\alpha-1}}$. Then to prove Theorem 3, it suffices to extend this result to the following: for any $2 \leq i \leq d$,

$$\mathbb{P}(s_i < n^{\alpha-1}T_i^{(n)} \leq t_i | s_{i+1} < n^{\alpha-1}T_{i+1}^{(n)} \leq t_{i+1}, \dots, s_d < n^{\alpha-1}T_d^{(n)} \leq t_d)$$

converges to $(1 + \frac{s_i}{\alpha\Gamma(\alpha)})^{-\frac{1}{\alpha-1}} - (1 + \frac{t_i}{\alpha\Gamma(\alpha)})^{-\frac{1}{\alpha-1}}$. The idea is the same as in the case $d = 2$.

We split the event $\{s_{i+1} < n^{\alpha-1}T_{i+1}^{(n)} \leq t_{i+1}, \dots, s_d < n^{\alpha-1}T_d^{(n)} \leq t_d\}$ into several classes. In each class, every individual from $i+1, \dots, d$ is specified whether it encounters a binary collision or it is implied in a multiple collision of $\Pi^{(d+1,n)}$. It is easy to see that there are 2^{d-i} disjoint classes which we denote by $\Delta_1, \dots, \Delta_{2^{d-i}}$:

$$\{s_{i+1} < n^{\alpha-1}T_{i+1}^{(n)} \leq t_{i+1}, \dots, s_d < n^{\alpha-1}T_d^{(n)} \leq t_d\} = \bigcup_{p=1}^{2^{d-i}} \Delta_p.$$

Then we have

$$\begin{aligned} & \mathbb{P}(s_i < n^{\alpha-1}T_1^{(n)} \leq t_i | s_{i+1} < n^{\alpha-1}T_{i+1}^{(n)} \leq t_{i+1}, \dots, s_d < n^{\alpha-1}T_d^{(n)} \leq t_d) \\ &= \mathbb{P}(s_i < n^{\alpha-1}T_1^{(n)} \leq t_i | \bigcup_{p=1}^{2^{d-i}} \Delta_p) \\ &= \frac{\sum_{p=1}^{2^{d-i}} \mathbb{P}(s_i < n^{\alpha-1}T_1^{(n)} \leq t_i, \Delta_p)}{\mathbb{P}(\bigcup_{p=1}^{2^{d-i}} \Delta_p)} \\ &= \frac{\sum_{p=1}^{2^{d-i}} \mathbb{P}(\Delta_p) \mathbb{P}(s_i < n^{\alpha-1}T_1^{(n)} \leq t_i | \Delta_p)}{\mathbb{P}(\bigcup_{p=1}^{2^{d-i}} \Delta_p)}. \end{aligned}$$

For each $\mathbb{P}(s_i < n^{\alpha-1}T_1^{(n)} \leq t_i | \Delta_p)$, $1 \leq p \leq 2^{d-i}$, the method is in the case $d = 2$, and the details are left to the reader. Together with Lemma 1, this gives equality (5), hence Theorem 3.

4. PROOFS OF PROPOSITION 4

In this Section, we give the proof of Proposition 4. First of all, we give 2 technical lemmas.

Lemma 2. *Let $(X_i)_{i \geq 1}, X$ be random non-negative variables such that $(X_i)_{i \geq 1}$ converges in distribution to X . Assume that there exists $\beta > 1$, and $C > 0$, such that for all $i \geq 1$, $\mathbb{E}[(X_i)^\beta] \leq C$, then $\mathbb{E}[X_i]$ converges to $\mathbb{E}[X]$.*

Proof. Thanks to Skorohod's representation theorem (see [3] page 70), we can construct random variables $(Y_i)_{i \geq 1}, Y$ such that for any $i \in \mathbb{N}$, $Y_i \stackrel{d}{=} X_i$, $Y \stackrel{d}{=} X$ and $(Y_i)_{i \geq 1}$ converges almost surely to Y . By Fatou's Lemma, $\mathbb{E}[X^\beta] = \mathbb{E}[Y^\beta] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[(Y_i)^\beta] \leq C$.

We use the decomposition: $Y_i = Y_i \mathbf{1}_{\{Y_i \leq Y+1\}} + Y_i \mathbf{1}_{\{Y_i > Y+1\}}$.

For the first part, $\mathbb{E}[Y_i \mathbf{1}_{\{Y_i \leq Y+1\}}]$ converges to $\mathbb{E}[Y]$ by the dominated convergence theorem.

For the second part, using Hölder's inequality, we have

$$\begin{aligned}\mathbb{E}[Y_i \mathbf{1}_{\{Y_i > Y+1\}}] &\leq \mathbb{E}[Y_i^\beta] (\mathbb{E}[\mathbf{1}_{\{Y_i > Y+1\}}])^{\frac{\beta-1}{\beta}} \\ &= \mathbb{E}[Y_i^\beta] (\mathbb{P}(\{Y_i > Y+1\}))^{\frac{\beta-1}{\beta}} \\ &\leq C (\mathbb{P}(\{Y_i > Y+1\}))^{\frac{\beta-1}{\beta}},\end{aligned}$$

which converges to 0.

Then $\mathbb{E}[Y_i]$ converges to $\mathbb{E}[Y]$. This achieves the proof. \square

Lemma 3. *Let $a > 0, b > 0, \beta > 2$. Then*

$$0 < (a+b)^\beta \leq a^\beta + b^\beta + \beta 2^{\beta-1} a b^{\beta-1} + \beta 2^{\beta-1} b a^{\beta-1}.$$

Proof. If $0 \leq m \leq 1$, then

$$(1+m)^\beta \leq 1 + \beta 2^{\beta-1} m \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

We have used that the function $m \mapsto (1+m)^\beta$ is convex and that $\beta 2^{\beta-1}$ is the derivative of $(1+m)^\beta$ at $m = 1$.

If $1 < m$, then

$$(1+m)^\beta = m^\beta (1 + \frac{1}{m})^\beta \leq (m)^\beta (1 + \beta 2^{\beta-1} \frac{1}{m}) \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

Hence for all $m > 0$,

$$(1+m)^\beta \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

Then for all $a > 0, b > 0$,

$$\begin{aligned}(a+b)^\beta &= a^\beta (1 + \frac{b}{a})^\beta \\ &\leq a^\beta (1 + (\frac{b}{a})^\beta + \beta 2^{\beta-1} \frac{b}{a} + \beta 2^{\beta-1} (\frac{b}{a})^{\beta-1}) \\ &= a^\beta + b^\beta + \beta 2^{\beta-1} a b^{\beta-1} + \beta 2^{\beta-1} b a^{\beta-1}.\end{aligned}$$

This achieves the proof. \square

We are now able to prove Proposition 4. We first prove the one-dimensional result.

Proof of the first part of Proposition 4:

Recall that T is a random variable with density function

$$f_T(x) = \frac{1}{(\alpha-1)\Gamma(\alpha)} (1 + \frac{x}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}-1} \mathbf{1}_{x \geq 0}.$$

We prove here for β being a positive real number.

- (1) If $\beta < \frac{\alpha}{\alpha-1}$, then $\lim_{n \rightarrow \infty} \mathbb{E}[(n^{\alpha-1} T_1^{(n)})^\beta] = \mathbb{E}[T^\beta]$.
- (2) If $\beta \geq \frac{\alpha}{\alpha-1}$, then $\lim_{n \rightarrow \infty} \mathbb{E}[(n^{\alpha-1} T_1^{(n)})^\beta] = \infty$.

In the case $\beta \geq \frac{\alpha}{\alpha-1}$, one gets $\mathbb{E}[T^\beta] = \infty$. Theorem 3 tells that $(n^{\alpha-1}T_1^{(n)})^\beta$ converges in distribution to T^β , so we deduce that $\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta]$ converges to ∞ .

If $0 \leq \beta < \frac{\alpha}{\alpha-1}$, according to Lemma 2, it suffice to prove that $\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta]$ is bounded for all n . Notice that given $0 < \beta_1 < \beta_2$, then $\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta_1}] \leq (\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta_2}])^{\frac{\beta_1}{\beta_2}}$ which means that, if $\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta_2}]$ is bounded for all n , the same for $\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta_1}]$. So we deduce that it suffices to consider $2 \leq \beta < \frac{\alpha}{\alpha-1}$. We will first prove that $(\mathbb{E}[n^{\alpha-1}T_1^{(n)}], n \geq 2)$ is bounded, and then that $(\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta], n \geq 2)$ is bounded.

Step 1: $(\mathbb{E}[n^{\alpha-1}T_1^{(n)}], n \geq 2)$ is bounded.

First of all, we remark that for fixed $n \geq 2$, we have $\mathbb{E}[n^{\alpha-1}T_1^{(n)}] < \infty$. We prove by induction on n that if $C_1 > 0$ is a constant large enough, then for all $n \geq 2$, we have $\mathbb{E}[n^{\alpha-1}T_1^{(n)}] < C_1$.

Let $n \geq 3$. We assume that there exists $C_1 > 0$ such that for all $2 \leq k \leq n-1$, we have $\mathbb{E}[k^{\alpha-1}T_1^{(k)}] \leq C_1$. Writing the decomposition of $T_1^{(n)}$ at the first collision, we have

$$n^{\alpha-1}T_1^{(n)} = n^{\alpha-1} \left(\frac{e_0}{g_n} + \sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)} \right),$$

where $H_k := \{\text{we have } k \text{ individuals after the first coalescence, and individual 1 is not involved in this collision.}\}$, e_0 is a unit exponential random variable, $\bar{T}_1^{(k)} \stackrel{d}{=} T_1^{(k)}$, and all these random variables $e_0, g_n, \bar{T}_1^{(k)}, \mathbf{1}_{\{H_k\}}$ are independent. We have

$$\begin{aligned} \mathbb{E}[n^{\alpha-1}T_1^{(n)}] &= \mathbb{E}[n^{\alpha-1}(\frac{e_0}{g_n} + \sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)})] \\ &= n^{\alpha-1} \mathbb{E}[\frac{e_0}{g_n}] + n^{\alpha-1} \sum_{k=2}^{n-1} \mathbb{P}(H_k) \mathbb{E}[\bar{T}_1^{(k)}] \\ &\leq \frac{n^{\alpha-1}}{g_n} + C_1 n^{\alpha-1} \sum_{k=2}^{n-1} \mathbb{P}(H_k) k^{1-\alpha} \\ &= \frac{n^{\alpha-1}}{g_n} + C_1 \sum_{k=2}^{n-1} \mathbb{P}(H_k) \left(\frac{n}{k}\right)^{\alpha-1} \\ &\leq \frac{n^{\alpha-1}}{g_n} + C_1 \sum_{k=2}^{n-1} \mathbb{P}(H_k) \frac{n}{k} \end{aligned}$$

Remark that $n - k + 1$ given individuals among n coalesce at rate $\lambda_{n,n-k+1}$. Hence

$$\mathbb{P}(H_k) = \frac{\binom{n-1}{n-k+1} \lambda_{n,n-k+1}}{g_n} = \frac{\int_0^1 \binom{n-1}{k-2} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n}.$$

We then have

$$\begin{aligned}
& \mathbb{E}[n^{\alpha-1}T_1^{(n)}] \\
&= \frac{n^{\alpha-1}}{g_n} + C_1 \sum_{k=2}^{n-1} \frac{\int_0^1 \binom{n-1}{k-2} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} \frac{n}{k} \\
&= \frac{n^{\alpha-1}}{g_n} + C_1 \sum_{k=2}^{n-1} \frac{\int_0^1 \binom{n-1}{k-2} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} \left(\frac{n}{k-1} - \frac{n}{k(k-1)} \right) \\
&= \frac{n^{\alpha-1}}{g_n} + C_1 \left(\frac{\sum_{k=2}^{n-1} \int_0^1 \binom{n}{k-1} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} \right. \\
&\quad \left. - \frac{1}{n+1} \frac{\sum_{k=2}^{n-1} \int_0^1 \binom{n+1}{k} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} \right) \\
&\leq \frac{n^{\alpha-1}}{g_n} + C_1 \left(1 - \frac{1}{n+1} \frac{\int_0^1 \sum_{k=2}^{n-1} \binom{n+1}{k} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} \right).
\end{aligned}$$

For a technical purpose, we define $\nu^*(dx) := \frac{1}{1-x} \nu(dx)$, $\Pi^{(n,*)}$ the n -coalescent process associated with $\nu^*(dx)$, and $g_n^* = \int_0^1 \sum_{a=2}^n \binom{n}{a} x^a (1-x)^{n-a} \nu^*(dx)$. We have

$$\begin{aligned}
& \frac{\int_0^1 \sum_{k=2}^{n-1} \binom{n+1}{k} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} \\
&= \frac{\int_0^1 \sum_{k=2}^{n-1} \binom{n+1}{k} x^{n-k+1} (1-x)^k \frac{1}{1-x} \nu(dx)}{g_n} \\
&= \frac{\int_0^1 \sum_{k=2}^{n-1} \binom{n+1}{k} x^{n-k+1} (1-x)^k \nu^*(dx)}{g_n} \\
&= \frac{g_{n+1}^*}{g_n} - \frac{\int_0^1 \binom{n+1}{1} x^n (1-x) \nu^*(dx) + \int_0^1 x^{n+1} \nu^*(dx)}{g_n}.
\end{aligned}$$

Using Lemma 2.2 of [10], we have $g_n = \frac{1}{\alpha\Gamma(\alpha)} n^\alpha + O(n^{\alpha-1})$. Moreover, while $\frac{\nu^*(dx)}{dx} = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} x^{-1-\alpha} + O(x^{-\alpha})$ when x tends to 0, then the n -coalescent $\Pi^{(n,*)}$ also satisfies the assumptions of Lemma 2.2 of [10]. Hence we get $g_n^* = \frac{1}{\alpha\Gamma(\alpha)} n^\alpha + O(n^{\alpha-1})$. Using again that $\frac{\nu^*(dx)}{dx} = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} x^{-1-\alpha} + O(x^{-\alpha})$, we get $\int_0^1 \binom{n+1}{1} x^n (1-x) \nu^*(dx) = O(n^{-1})$ and $\int_0^1 x^{n+1} \nu^*(dx) = O(n^{-1})$, and hence

$$(7) \quad \lim_{n \rightarrow \infty} \frac{\int_0^1 \sum_{k=2}^{n-1} \binom{n+1}{k} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} = 1.$$

We have

$$\begin{aligned}
\mathbb{E}[n^{\alpha-1}T_1^{(n)}] &\leq \frac{n^{\alpha-1}}{g_n} + C_1 \left(1 - \frac{1}{n+1} \frac{\int_0^1 \sum_{k=2}^{n-1} \binom{n+1}{k} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} \right) \\
&= C_1 + \frac{1}{n} \left(\frac{n^\alpha}{g_n} - C_1 \frac{n}{n+1} \frac{\int_0^1 \sum_{k=2}^{n-1} \binom{n+1}{k} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} \right).
\end{aligned}$$

Using (7) and that $\frac{n^\alpha}{g_n}$ is bounded, we get that there exists $C_0 > 0$ such that, if $C_1 \geq C_0$,

$$\frac{n^\alpha}{g_n} - C_1 \frac{n}{n+1} \frac{\int_0^1 \sum_{k=2}^{n-1} \binom{n+1}{k} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} < 0$$

for all $n \geq 2$. Hence, if C_1 is also greater than $\mathbb{E}[2^{\alpha-1}T_1^{(2)}]$, we have proved by induction that $(\mathbb{E}[n^{\alpha-1}T_1^{(n)}], n \geq 2)$ is bounded by C_1 .

Remark 5. In the above proof, we get that $\sum_{k=2}^{n-1} \mathbb{P}(H_k) \frac{n}{k} = 1 + O(n^{-1})$. This estimation is useful for later use.

Step 2: $(\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta], n \geq 2)$ is bounded.

In this step, we employ the method used previously. First of all, we remark that for fixed $n \geq 2$, we have $\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta] < \infty$. Let $n \geq 3$. We assume that there exists $C_2 > 0$ such that for all $2 \leq k \leq n-1$, we have $\mathbb{E}[(k^{\alpha-1}T_1^{(k)})^\beta] \leq C_2$. With the notations of the first step, and using Lemma 3, we have

$$\begin{aligned}
&\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta] \\
&= \mathbb{E}[(n^{\alpha-1}(\frac{e_0}{g_n} + \sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)}))^\beta] \\
&\leq n^{(\alpha-1)\beta} \mathbb{E}[(\frac{e_0}{g_n})^\beta] + n^{(\alpha-1)\beta} \mathbb{E}[(\sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)})^\beta] \\
&\quad + n^{(\alpha-1)\beta} \mathbb{E}[\beta 2^{\beta-1} \frac{e_0}{g_n} (\sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)})^{\beta-1}] + n^{(\alpha-1)\beta} \mathbb{E}[\beta 2^{\beta-1} (\frac{e_0}{g_n})^{\beta-1} \sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)}] \\
(8) \quad &= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}
\end{aligned}$$

where

$$\begin{aligned}
I_{n,1} &= n^{(\alpha-1)\beta} \mathbb{E}\left[\left(\frac{e_0}{g_n}\right)^\beta\right], \\
I_{n,2} &= n^{(\alpha-1)\beta} \mathbb{E}\left[\left(\sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)}\right)^\beta\right], \\
I_{n,3} &= n^{(\alpha-1)\beta} \mathbb{E}\left[\beta 2^{\beta-1} \frac{e_0}{g_n} \left(\sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)}\right)^{\beta-1}\right], \\
I_{n,4} &= n^{(\alpha-1)\beta} \mathbb{E}\left[\beta 2^{\beta-1} \left(\frac{e_0}{g_n}\right)^{\beta-1} \sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)}\right].
\end{aligned}$$

Recall that $g_n \sim \frac{1}{\alpha\Gamma(\alpha)} n^\alpha$. Hence there exists a constant $K_1 > 0$ such that for all $n \geq 2$,

$$(9) \quad I_{n,1} \leq \frac{K_1}{n^{\alpha\beta}} \leq \frac{K_1}{n},$$

because $\beta \geq 2 > 1$.

To bound $I_{n,2}$, we introduce H_0 the event that individual 1 is involved in the first collision. Remark that $H_0 = (\bigcup_{k=2}^{n-1} H_k)^c$. Recall that $Y_1^{(n)}$ is the number of individuals right after the first coalescence and $X_1^{(n)} = n - Y_1^{(n)}$ is number of individuals lost during the first coalescence. Then

$$\begin{aligned}
\mathbb{P}(H_0) &= 1 - \mathbb{P}(\text{individual 1 is not coalesced at the first coalescence}) \\
&= 1 - \mathbb{E}[\mathbb{P}(\text{individual 1 is not coalesced at the first coalescence} | X_1^{(n)})] \\
&= 1 - \mathbb{E}\left[\frac{\binom{n-1}{X_1^{(n)}+1}}{\binom{n}{X_1^{(n)}+1}}\right] \\
&= \mathbb{E}\left[\frac{X_1^{(n)} + 1}{n}\right].
\end{aligned}$$

Recall the following properties of $X_1^{(n)}$ (see [10], page 1003). The law of $X_1^{(n)}$ is given by: if $\rho(t) = \nu([t, 1])$, for any $1 \leq k \leq n-1$,

$$(10) \quad \mathbb{P}(X_1^{(n)} \geq k) = \frac{(n-2)!}{(k)!(n-k-1)!} \frac{\int_0^1 (1-t)^{n-k-1} t^k \rho(t) dt}{\int_0^1 (1-t)^{n-2} t \rho(t) dt}.$$

When n tends to ∞ , its first two moments satisfy $\mathbb{E}[X_1^{(n)}] = \frac{1}{\alpha-1} + O(n^{1-\alpha})$ (see Lemma 2.3 in [10]) and $n^{2-\alpha} \mathbb{E}[(X_1^{(n)})^2]$ is bounded (see Lemma 2.4 in [10]).

Hence we get that $\mathbb{P}(H_0) = \frac{\alpha}{n(\alpha-1)} + O(n^{-\alpha})$.

Recall that $\sigma_1^{(n)}$ is the number of collisions which occur until individual 1 is concerned. Using the asymptotics of $\mathbb{P}(H_0)$ and of the two first moments of $X_1^{(n)}$, we have:

$$(11) \quad \mathbb{E}[X_1^{(n)} | (H_0)^c] = \frac{\mathbb{E}[X_1^{(n)} \mathbf{1}_{\{\sigma_1^{(n)} > 1\}}]}{\mathbb{P}((H_0)^c)} \leq \frac{\mathbb{E}[X_1^{(n)}]}{\mathbb{P}((H_0)^c)} = \frac{1}{\alpha-1} + O(n^{1-\alpha})$$

and

$$(12) \quad \mathbb{E}[(X_1^{(n)})^2/n^2|(H_0)^c] = \frac{\mathbb{E}[(X_1^{(n)})^2/n^2 \mathbf{1}_{\{\sigma_1^{(n)} > 1\}}]}{\mathbb{P}((H_0)^c)} \leq \frac{\mathbb{E}[(X_1^{(n)})^2/n^2]}{\mathbb{P}((H_0)^c)} = O(n^{-\alpha}).$$

Let us fix $t \in (0, 1)$. Recall that $\Pi^{(n,*)}$ is the n -coalescent process associated with $\nu^*(dx) = \frac{1}{1-x}\nu(dx)$, and define $X_1^{(n,*)}$ as the number of individuals we lose at first jump, and $\rho^*(s) = \nu^*([s, 1])$ for any $s \in (0, 1]$. Using the assumption on $\mathbb{E}[(T_1^{(k)})^\beta]$, we have

$$\begin{aligned} I_{n,2} &= n^{(\alpha-1)\beta} \mathbb{E}\left[\left(\sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)}\right)^\beta\right] \\ &\leq n^{(\alpha-1)\beta} \sum_{k=2}^{n-1} \mathbb{P}(H_k) \frac{C_2}{k^{\beta(\alpha-1)}} = C_2 \sum_{k=2}^{n-1} \mathbb{P}(H_k) \left(\frac{n}{k}\right)^{(\alpha-1)\beta} \\ &= C_2 \sum_{k=2}^{\lfloor nt \rfloor} \mathbb{P}(H_k) \left(\frac{n}{k}\right)^{(\alpha-1)\beta} + C_2 \sum_{k=\lfloor nt \rfloor + 1}^{n-1} \mathbb{P}(H_k) \left(\frac{n}{k}\right)^{(\alpha-1)\beta} \\ &= C_2 \sum_{k=2}^{\lfloor nt \rfloor} \mathbb{P}(H_k) \left(\frac{n}{k}\right)^{(\alpha-1)\beta} + C_2 \mathbb{P}((H_0)^c) \left(\mathbb{E}\left[\left(\frac{n}{Y_1^{(n)}}\right)^{(\alpha-1)\beta} \mathbf{1}_{\{Y_1^{(n)} \geq \lfloor nt \rfloor + 1\}}\right] | (H_0)^c \right). \end{aligned}$$

We study these two terms one by one.

$$\begin{aligned} \sum_{k=2}^{\lfloor nt \rfloor} \mathbb{P}(H_k) \left(\frac{n}{k}\right)^{(\alpha-1)\beta} &\leq \sum_{k=2}^{\lfloor nt \rfloor} \mathbb{P}(H_k) \left(\frac{n}{k}\right)^2 \\ &\leq \sum_{k=2}^{\lfloor nt \rfloor} \frac{\int_0^1 \binom{n-1}{k-2} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} \frac{n}{k-1} \frac{n+1}{k} \\ &= \sum_{k=2}^{\lfloor nt \rfloor} \frac{\int_0^1 \binom{n+1}{k} x^{n-k+1} (1-x)^k \nu^*(dx)}{g_n} \\ &= \frac{g_{n+1}^*}{g_n} \sum_{k=2}^{\lfloor nt \rfloor} \frac{\int_0^1 \binom{n+1}{k} x^{n-k+1} (1-x)^k \nu^*(dx)}{g_{n+1}^*} \\ &\leq \frac{g_{n+1}^*}{g_n} \mathbb{P}(X_1^{(n+1,*)} \geq n - \lfloor nt \rfloor). \end{aligned}$$

Using (10) for $X_1^{(n+1,*)}$ and $k = n - \lfloor nt \rfloor$, we have

$$\mathbb{P}(X_1^{(n+1,*)} \geq n - \lfloor nt \rfloor) = \frac{(n-1)!}{(n - \lfloor nt \rfloor)! (\lfloor nt \rfloor)!} \frac{\int_0^1 (1-s)^{\lfloor nt \rfloor} s^{n - \lfloor nt \rfloor} \rho^*(s) ds}{\int_0^1 (1-s)^{n-1} s \rho^*(s) ds}.$$

Recall that $\frac{\nu^*(ds)}{ds} = \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)} s^{-1-\alpha} + O(s^{-\alpha})$ when s tends to 0. Hence $\rho^*(s) = \nu^*([s, 1]) = \frac{s^{-\alpha}}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)} + O(s^{1-\alpha})$. Moreover $\Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z} (1 + \frac{1}{12z} + o(\frac{1}{z}))$ when z tends to ∞ .

Hence we see that there exists two positive constants $K_{2,1}$ and $K_{2,2}$ (independent of n , but may depend on t) such that

$$\begin{aligned} \mathbb{P}(X_1^{(n,*)} \geq n - \lfloor nt \rfloor) &\leq K_{2,1} \frac{(n-1)!}{(n - \lfloor nt \rfloor)! (\lfloor nt \rfloor)!} \frac{\int_0^1 (1-s)^{\lfloor nt \rfloor} s^{n - \lfloor nt \rfloor} s^{-\alpha} ds}{\int_0^1 (1-s)^{n-1} s s^{-\alpha} ds} \\ &= K_{2,1} \frac{(n-1)!}{(n - \lfloor nt \rfloor)! (\lfloor nt \rfloor)!} \frac{\Gamma(\lfloor nt \rfloor + 1) \Gamma(n - \lfloor nt \rfloor - \alpha + 1)}{\Gamma(n) \Gamma(2 - \alpha)} \\ &\leq K_{2,2} n^{-\alpha}. \end{aligned}$$

Moreover, $\lim_{n \rightarrow \infty} \frac{g_{n+1}^*}{g_n} = 1$, then there exists a constant $K_{2,3} > 0$ (independent of n , but may depend on t), such that

$$(13) \quad \sum_{k=2}^{\lfloor nt \rfloor} \mathbb{P}(H_k) \left(\frac{n}{k}\right)^{(\alpha-1)\beta} \leq K_{2,3} n^{-\alpha}.$$

Let us now consider the second term. There exists a constant $K_{2,4} > 0$ such that if $u \in (0, 1-t)$, then

$$(1-u)^{(1-\alpha)\beta} \leq 1 + (\alpha-1)\beta u + K_{2,4} u^2.$$

$K_{2,4}$ depends on t .

Hence

$$\begin{aligned} &\mathbb{E}\left[\left(\frac{n}{Y_1^{(n)}}\right)^{(\alpha-1)\beta} \mathbf{1}_{\{Y_1^{(n)} \geq \lfloor nt \rfloor + 1\}} | (H_0)^c\right] \\ &= \mathbb{E}\left[\left(\frac{1}{1 - \frac{X_1^{(n)}}{n}}\right)^{(\alpha-1)\beta} \mathbf{1}_{\{1 \leq X_1^{(n)} \leq n - \lfloor nt \rfloor - 1\}} | (H_0)^c\right] \\ &\leq \mathbb{E}\left[(1 + (\alpha-1)\beta \frac{X_1^{(n)}}{n} + K_{2,4} (\frac{X_1^{(n)}}{n})^2) \mathbf{1}_{\{1 \leq X_1^{(n)} \leq n - \lfloor nt \rfloor - 1\}} | (H_0)^c\right] \\ &\leq \mathbb{E}\left[(1 + (\alpha-1)\beta \frac{X_1^{(n)}}{n} + K_{2,4} (\frac{X_1^{(n)}}{n})^2) | (H_0)^c\right] \\ &\leq 1 + \frac{\beta}{n} + K_{2,5} n^{-\alpha}, \end{aligned}$$

for a certain constant $K_{2,5}$. We refer to (11) and (12) for the last inequality.

Recall that $\mathbb{P}(H_0) = \frac{\alpha}{n(\alpha-1)} + O(n^{-\alpha})$ and $\beta < \frac{\alpha}{\alpha-1}$. Hence there exists $K_2 > 0$ and $N_1 > 0$ such that for $n \geq N_1$,

$$\begin{aligned} I_{n,2} &\leq C_2 \sum_{k=2}^{\lfloor nt \rfloor} \mathbb{P}(H_k) \left(\frac{n}{k}\right)^{(\alpha-1)\beta} + C_2 \mathbb{P}((H_0)^c) \left(\mathbb{E}\left[\left(\frac{n}{Y_1^{(n)}}\right)^{(\alpha-1)\beta} \mathbf{1}_{\{Y_1^{(n)} \geq \lfloor nt \rfloor + 1\}} | (H_0)^c\right] \right) \\ (14) \quad &\leq C_2 \left(1 - \frac{K_2}{n}\right). \end{aligned}$$

We now proceed to $I_{n,3}$. Notice that

$$\begin{aligned}\mathbb{E}[(T_1^{(k)})^{\beta-1}] &= k^{-(\alpha-1)(\beta-1)} \mathbb{E}[(k^{\alpha-1} T_1^{(k)})^{\beta-1}] \\ &\leq k^{-(\alpha-1)(\beta-1)} (\mathbb{E}[(k^{\alpha-1} T_1^{(k)})^\beta])^{\frac{\beta-1}{\beta}} \\ &\leq k^{-(\alpha-1)(\beta-1)} (C_2)^{\frac{\beta-1}{\beta}}.\end{aligned}$$

Hence we have

$$\begin{aligned}I_{n,3} &= n^{(\alpha-1)\beta} \mathbb{E}[\beta 2^{\beta-1} \frac{e_0}{g_n} \sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} (\bar{T}_1^{(k)})^{\beta-1}] \\ &= n^{(\alpha-1)\beta} \beta 2^{\beta-1} \frac{1}{g_n} \sum_{k=2}^{n-1} \mathbb{P}(H_k) \mathbb{E}[(\bar{T}_1^{(k)})^{\beta-1}] \\ &\leq (C_2)^{\frac{\beta-1}{\beta}} n^{(\alpha-1)\beta} \beta 2^{\beta-1} \frac{1}{g_n} \sum_{k=2}^{n-1} \mathbb{P}(H_k) k^{-(\alpha-1)(\beta-1)} \\ &= (C_2)^{\frac{\beta-1}{\beta}} n^{(\alpha-1)\beta} \beta 2^{\beta-1} \frac{1}{g_n} \sum_{k=2}^{n-1} \mathbb{P}(H_k) \left(\frac{n}{k}\right)^{(\alpha-1)(\beta-1)} \\ &\leq (C_2)^{\frac{\beta-1}{\beta}} n^{(\alpha-1)\beta} \beta 2^{\beta-1} \frac{1}{g_n} \sum_{k=2}^{n-1} \mathbb{P}(H_k) \frac{n}{k}.\end{aligned}$$

Recall (see Remark 5) that $\mathbb{E}[\sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \frac{n}{k}] = 1 + O(n^{-1})$, and that $g_n \sim \frac{1}{\alpha\Gamma(\alpha)} n^\alpha$. Hence there exists a constant $K_3 > 0$ such that

$$(15) \quad I_{n,3} \leq \frac{(C_2)^{\frac{\beta-1}{\beta}} K_3}{n}.$$

We now conclude with $I_{n,4}$. Recall that by step 1, there exists $C_1 > 0$ such that for all $n \geq 2$, $\mathbb{E}[n^{\alpha-1} T_1^{(n)}] \leq C_1$. Then

$$\begin{aligned}I_{n,4} &= n^{(\alpha-1)\beta} \mathbb{E}[\beta 2^{\beta-1} (\frac{e_0}{g_n})^{\beta-1} \sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)}] \\ &= \beta 2^{\beta-1} \mathbb{E}[e_0^{\beta-1}] (g_n)^{1-\beta} \mathbb{E}[\sum_{k=2}^{n-1} \mathbf{1}_{\{H_k\}} \bar{T}_1^{(k)}] \\ &\leq \beta 2^{\beta-1} \mathbb{E}[e_0^{\beta-1}] (g_n)^{1-\beta} \mathbb{E}[T_1^{(n)}] \\ &\leq C_1 \beta 2^{\beta-1} \mathbb{E}[e_0^{\beta-1}] n^{1-\alpha\beta}.\end{aligned}$$

Hence, there exists K_4 such that

$$(16) \quad I_{n,4} \leq \frac{K_4}{n^{\alpha\beta-1}} \leq \frac{K_4}{n}.$$

In the last inequality, we use the fact that $\beta \geq 2$.

Using (8),(9),(14),(15),(16), we have proved that if there exists $C_2 > 0$ such that for all $2 \leq k \leq n-1$, $\mathbb{E}[(k^{\alpha-1} T_1^{(k)})^\beta] \leq C_2$, then for any $n \geq N_1$

$$\mathbb{E}[(n^{\alpha-1} T_1^{(n)})^\beta] \leq C_2 + (K_1 - C_2 K_2 + (C_2)^{\frac{\beta-1}{\beta}} K_3 + K_4) \frac{1}{n}.$$

If we choose C_2 large enough such that $\mathbb{E}[(k^{\alpha-1}T_1^{(k)})^\beta] \leq C_2$ for $2 \leq k \leq N_1$ and $K_1 - C_2K_2 + (C_2)^{\frac{\beta-1}{\beta}}K_3 + K_4 \leq 0$, we have proved by induction that $(\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta], n \geq 2)$ is bounded by C_2 . The proof of the first part of Proposition 4 is achieved.

Proof of the second part of Proposition 4:

Here, we prove that for fixed $d \in \mathbb{N}$, and $\beta_1, \beta_2, \dots, \beta_d$ non-negative real numbers such that $\sum_{i=1}^d \beta_i < \frac{\alpha}{\alpha-1}$, we have

$$\lim_{n \rightarrow \infty} \mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta_1} (n^{\alpha-1}T_2^{(n)})^{\beta_2} \dots (n^{\alpha-1}T_d^{(n)})^{\beta_d}] = \mathbb{E}[T^{\beta_1}] \mathbb{E}[T^{\beta_2}] \dots \mathbb{E}[T^{\beta_d}].$$

For any positive $\beta_i, 1 \leq i \leq d$ such that $\sum_{i=1}^d \beta_i < \frac{\alpha}{\alpha-1}$, by Hölder's inequality,

$$\begin{aligned} & \mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta_1} \dots (n^{\alpha-1}T_d^{(n)})^{\beta_d}] \\ & \leq (\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\sum_{i=1}^d \beta_i}])^{\frac{\beta_1}{\sum_{i=1}^d \beta_i}} \dots (\mathbb{E}[(n^{\alpha-1}T_d^{(n)})^{\sum_{i=1}^d \beta_i}])^{\frac{\beta_d}{\sum_{i=1}^d \beta_i}} \\ & = \mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\sum_{i=1}^d \beta_i}] \rightarrow \mathbb{E}[T^{\sum_{i=1}^d \beta_i}], \end{aligned}$$

when n tends to ∞ .

So $\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta_d} \dots (n^{\alpha-1}T_d^{(n)})^{\beta_d}]$ is bounded for all $n \geq d$. Furthermore, Theorem 3 gives that

$$(n^{\alpha-1}T_1^{(n)})^{\beta_d} \dots (n^{\alpha-1}T_d^{(n)})^{\beta_d}$$

converges in distribution to $(T_1)^{\beta_1} \dots (T_d)^{\beta_d}$.

We conclude with Lemma 2 to get Corollary 3.

5. PROOF OF THEOREM 1 AND COROLLARY 2

First of all, let us give the proof of Theorem 1. Recall that $L_{ext}^{(n)} = \sum_{k=1}^n T_k^{(n)}$.

$$\begin{aligned} & \mathbb{E}[(n^{\alpha-2}L_{ext}^{(n)} - \alpha(\alpha-1)\Gamma(\alpha))^2] \\ & = \mathbb{E}[(\frac{\sum_{k=1}^n n^{\alpha-1}T_k^{(n)}}{n} - \alpha(\alpha-1)\Gamma(\alpha))^2] \\ & = \mathbb{E}[(\frac{\sum_{k=1}^n n^{\alpha-1}T_k^{(n)}}{n} - \mathbb{E}[n^{\alpha-1}T_1^{(n)}] + \mathbb{E}[n^{\alpha-1}T_1^{(n)}] - \alpha(\alpha-1)\Gamma(\alpha))^2] \\ & = \mathbb{E}[(\frac{\sum_{k=1}^n n^{\alpha-1}T_k^{(n)}}{n} - \mathbb{E}[n^{\alpha-1}T_1^{(n)}])^2] + (\mathbb{E}[n^{\alpha-1}T_1^{(n)}] - \alpha(\alpha-1)\Gamma(\alpha))^2 \\ & = \frac{Var(n^{\alpha-1}T_1^{(n)})}{n} + \frac{n(n-1)}{n^2}Cov(n^{\alpha-1}T_1^{(n)}, n^{\alpha-1}T_2^{(n)}) + (\mathbb{E}[n^{\alpha-1}T_k^{(n)}] - \alpha(\alpha-1)\Gamma(\alpha))^2. \end{aligned}$$

The first part of Theorem 4 implies that $Var(n^{\alpha-1}T_1^{(n)}) \rightarrow Var(T)$ and $\mathbb{E}[n^{\alpha-1}T_1^{(n)}] \rightarrow \mathbb{E}[T] = \alpha(\alpha-1)\Gamma(\alpha)$, when n tends to ∞ .

The second part of Theorem 4 with $d = 2$ implies that $Cov(n^{\alpha-1}T_1^{(n)}, n^{\alpha-1}T_2^{(n)})$ converges to 0 as n tends to ∞ .

Hence $\mathbb{E}[(n^{\alpha-2}L_{ext}^{(n)} - \alpha(\alpha-1)\Gamma(\alpha))^2]$ converges to 0. This ends the proof of Theorem 1.

The proof of Corollary 2 is straightforward. Recall that (see [2]), $n^{\alpha-2}L^{(n)}$ converges in probability to

$$\frac{\alpha\Gamma(\alpha)(\alpha-1)}{2-\alpha}.$$

Hence we get that $L_{ext}^{(n)}/L^{(n)}$ converges in probability to $2-\alpha$, which is Corollary 2.

REFERENCES

- [1] E. Arnason. Mitochondrial cytochrome b dna variation in the high-fecundity atlantic cod: trans-atlantic clines and shallow gene genealogy. *Genetics*, 166(4):1871, 2004.
- [2] J. Berestycki, N. Berestycki, and J. Schweinsberg. Beta-coalescents and continuous stable random trees. *Ann. Probab.*, 35(5):1835–1887, 2007.
- [3] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [4] M. Birkner, J. Blath, M. Capaldo, A. Etheridge, M. Möhle, J. Schweinsberg, and A. Wakolbinger. Alpha-stable branching and beta-coalescents. *Electron. J. Probab.*, 10:no. 9, 303–325 (electronic), 2005.
- [5] M. G. B. Blum and O. François. Minimal clade size and external branch length under the neutral coalescent. *Adv. in Appl. Probab.*, 37(3):647–662, 2005.
- [6] E. Bolthausen and A.-S. Sznitman. On Ruelle’s probability cascades and an abstract cavity method. *Comm. Math. Phys.*, 197(2):247–276, 1998.
- [7] J. Boom, E. Boulding, and A. Beckenbach. Mitochondrial DNA variation in introduced populations of Pacific oyster, *Crassostrea gigas*, in British Columbia. *Canadian journal of fisheries and aquatic sciences(Print)*, 51(7):1608–1614, 1994.
- [8] A. Bovier and I. Kurkova. Much ado about Derrida’s GREM. In *Spin glasses*, volume 1900 of *Lecture Notes in Math.*, pages 81–115. Springer, Berlin, 2007.
- [9] A. Caliebe, R. Neininger, M. Krawczak, and U. Roesler. On the length distribution of external branches in coalescence trees: genetic diversity within species. *Theoretical Population Biology*, 72(2):245–252, 2007.
- [10] J.-F. Delmas, J.-S. Dhersin, and A. Siri-Jégousse. Asymptotic results on the length of coalescent trees. *Ann. Appl. Probab.*, 18(2):997–1025, 2008.
- [11] J.-S. Dhersin, F. Freund, A. Siri-Jégousse, and L. Yuan. On the length of an external branch in the beta-coalescent. *Arxiv preprint arXiv:1201.3983*, 2012.
- [12] M. Drmota, A. Iksanov, M. Moehle, and U. Roesler. Asymptotic results concerning the total branch length of the Bolthausen-Sznitman coalescent. *Stochastic Process. Appl.*, 117(10):1404–1421, 2007.
- [13] M. Drmota, A. Iksanov, M. Moehle, and U. Roesler. A limiting distribution for the number of cuts needed to isolate the root of a random recursive tree. *Random Structures Algorithms*, 34(3):319–336, 2009.
- [14] R. Durrett. *Probability models for DNA sequence evolution*. Probability and its Applications (New York). Springer, New York, second edition, 2008.
- [15] B. Eldon and J. Wakeley. Coalescent processes when the distribution of offspring number among individuals is highly skewed. *Genetics*, 172:2621–2633, 2006.
- [16] F. Freund and M. Möhle. On the time back to the most recent common ancestor and the external branch length of the Bolthausen-Sznitman coalescent. *Markov Process. Related Fields*, 15(3):387–416, 2009.
- [17] Y. Fu and W. Li. Statistical tests of neutrality of mutations. *Genetics*, 133:693–709, 1993.
- [18] A. Gnedin, A. Iksanov, and M. Möhle. On asymptotics of exchangeable coalescents with multiple collisions. *J. Appl. Probab.*, 45(4):1186–1195, 2008.
- [19] A. Gnedin and Y. Yakubovich. On the number of collisions in Λ -coalescents. *Electron. J. Probab.*, 12:no. 56, 1547–1567 (electronic), 2007.
- [20] C. Goldschmidt and J. B. Martin. Random recursive trees and the Bolthausen-Sznitman coalescent. *Electron. J. Probab.*, 10:no. 21, 718–745 (electronic), 2005.
- [21] D. Hedgecock. Does variance in reproductive success limit effective population sizes of marine organisms? In *Genetics and Evolution of Aquatic Organisms*, pages 122–134. Springer, 1994. Edited by Beaumont, A.R.
- [22] S. Janson and G. Kersting. On the total external length of the kingman coalescent. *Electronic Journal of Probability*, 16:2203–2218, 2011.
- [23] G. Kersting. The asymptotic distribution of the length of beta-coalescent trees. *Arxiv*, 27(1), 2011.
- [24] M. Kimura. The number of heterozygous nucleotide sites maintained in a finite population due to steady flux of mutations. *Genetics*, 61(4):893–903, 1969.

- [25] J. F. C. Kingman. The coalescent. *Stochastic Process. Appl.*, 13(3):235–248, 1982.
- [26] J. F. C. Kingman. Origins of the Coalescent 1974-1982. *Genetics*, 156(4):1461–1463, 2000.
- [27] M. Möhle. On the number of segregating sites for populations with large family sizes. *Adv. in Appl. Probab.*, 38(3):750–767, 2006.
- [28] M. Möhle. Asymptotic results for coalescent processes without proper frequencies and applications to the two-parameter Poisson-Dirichlet coalescent. *Stochastic Process. Appl.*, 120(11):2159–2173, 2010.
- [29] J. Pitman. Coalescents with multiple collisions. *Ann. Probab.*, 27(4):1870–1902, 1999.
- [30] S. Sagitov. The general coalescent with asynchronous mergers of ancestral lines. *J. Appl. Probab.*, 36(4):1116–1125, 1999.
- [31] J. Schweinsberg. Coalescent processes obtained from supercritical Galton-Watson processes. *Stochastic Process. Appl.*, 106(1):107–139, 2003.

LAGA, INSTITUT GALILÉE, UNIVERSITÉ PARIS XIII, 93430 VILLETANEUSE, FRANCE
E-mail address: `dhersin@math.univ-paris13.fr`

LAGA, INSTITUT GALILÉE, UNIVERSITÉ PARIS XIII, 93430 VILLETANEUSE, FRANCE
E-mail address: `yuan@math.univ-paris13.fr`